LESSON 7.6 Inverse of a Square Matrix

We can't divide by a matrix so we need a different way to solve matrix equations.

Example $\frac{2}{3} \cdot \frac{3}{2}x = 9$

$\left(\frac{3}{2}\right)^{-1} = \frac{2}{3}$

$A^{-1} \cdot AX = B$ $\quad \frac{1}{5} \cdot 5 = 1$

$IX = A^{-1} \cdot B$

$X = A^{-1} \cdot B$

Definition of the Inverse of a Square Matrix

Let A be an $n \times n$ matrix and let I_n be the $n \times n$ identity matrix. If there exists a matrix A^{-1} such that

$$AA^{-1} = I_n = A^{-1}A$$

then A^{-1} is called the **inverse** of A. The symbol A^{-1} is read "A inverse."

If a matrix has an inverse it is called invertible. A non-square matrix can't have an inverse.

$$A^{-1} \cdot A = I_{n \times n} = A \cdot A^{-1}$$

Example

Show that A and B are inverses.

$$B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \quad A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$$

Solution

Show that $AB = I = BA$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example

If possible, find the inverse of the following matrix using your calculator.

$$C = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right] \rightarrow R_1 - R_2$$

Check-Point:

Verify that the following matrices are inverses.

$$A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{bmatrix}$$

$a = 3$
$b = -1$
$c = -2$
$d = 2$

$$A^{-1} = \frac{1}{6-2}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Finding an Inverse Matrix

$[A \vdots I]$

Let A be a square matrix of order n.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain $[A \ \vdots \ I]$.
2. If possible, row reduce A to I using elementary row operations on the *entire* matrix $[A \ \vdots \ I]$. The result will be the matrix $[I \ \vdots \ A^{-1}]$. If this is not possible, A is not invertible.
3. Check your work by multiplying to see that $AA^{-1} = I = A^{-1}A$.

Inverse of a 2x2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Determinant

A is invertible if and only if $ad - bc \neq 0$.

The inverse is given by:

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Solving Systems Using Inverses

$$\begin{cases} x + 2y - z = 4 \\ -x - y + 2z = 16 \\ 2x - y + 2z = 5 \end{cases} \longrightarrow \overset{A}{\begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 2 \\ 2 & -1 & 2 \end{bmatrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overset{B}{\begin{bmatrix} 4 \\ 16 \\ 5 \end{bmatrix}}$$

$$AX = B$$

$$\boxed{X = A^{-1}B}$$

A System of Equations with a Unique Solution

If A is an invertible matrix, the system of linear equations represented by $AX = B$ has a unique solution given by

$$X = A^{-1}B.$$

To solve the system, multiply the inverse of matrix A with the constant matrix, B. $A^{-1}B = X$.